REPRESENTATIONS AND COHOMOLOGY OF N-ARY MULTIPLICATIVE HOM-NAMBU-LIE ALGEBRAS

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Abstract. The aim of this paper is to provide cohomologies of n-ary Hom-Nambu-Lie algebras governing central extensions and one parameter formal deformations. We generalize to n-ary algebras the notions of derivations and representation introduced by Sheng for Hom-Lie algebras. Also we show that a cohomology of n-ary Hom-Nambu-Lie algebras could be derived from the cohomology of Hom-Leibniz algebras.

Introduction

The first instances of n-ary algebras in Physics appeared with a generalization of the Hamiltonian mechanics proposed in 1973 by Nambu [25]. More recent motivation comes from string theory and M-branes involving naturally an algebra with ternary operation called Bagger-Lambert algebra which give impulse to a significant development. It was used in [8] as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and an SO(8) R-symmetry that acts on the eight transverse scalars. On the other hand in the study of supergravity solutions describing M2-branes ending on M5-branes, the Lie algebra appearing in the original Nahm equations has to be replaced with a generalization involving ternary bracket in the lifted Nahm equations, see [9]. For other applications in Physics see [27], [28], [29].

The algebraic formulation of Nambu mechanics is du to Takhtajan [12, 31] while the abstract definition of n-ary Nambu algebras or n-ary Nambu-Lie algebras (when the bracket is skew symmetric) was given by Filippov in 1985 see [14]. The Leibniz n-ary algebras were introduced and studied in [11]. For deformation theory and cohomologies of n-ary algebras of Lie type, we refer to [4, 5, 15, 13, 31].

The general Hom-algebra structures arose first in connection to quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skew-symmetry and Jacobi conditions are twisted. For Hom-Lie algebras, Hom-associative algebras, Hom-Lie superalgebras, Hom-bialgebras ... see [1, 16, 19, 20, 21, 23]. Generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced in [6]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type such as n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type such as n-ary totally associative and n-ary partially associative algebras. See also [34, 35, 36].

In the first Section of this paper we summarize the definitions of n-ary Hom-Nambu (resp. Hom-Nambu-Lie) algebras and the multiplicative n-ary Hom-Nambu (resp. Hom-Nambu-Lie) algebras. In Section 2, we extend to n-ary algebras the notions of derivations and representation introduced for Hom-Lie algebras in [30]. In Section 3, we show that for an n-ary Hom-Nambu-Lie algebra $\mathcal{N}$, the space $\wedge^{n-1}\mathcal{N}$ carries a structure of Hom-Leibniz algebra. Section 4 is dedicated to central extensions. We provide a cohomology adapted to central extensions of n-ary multiplicative Hom-Nambu-Lie algebras. In Section 5, we provide a cohomology which is suitable for the study of one parameter formal deformations of n-ary Hom-Nambu-Lie algebras. In the last Section we show that the cohomology of n-ary Hom-Nambu-Lie algebras may be derived from the the cohomology of Hom-Leibniz algebras. To this end we generalize to twisted situation the process used by Daletskii and Takhtajan [12] for the classical case.

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1. The n-ary Hom-Nambu algebras

Throughout this paper, we will for simplicity of exposition assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

1.1. **Definitions.** In this section, we recall the definition of n-ary Hom-Nambu algebras and n-ary Hom-Nambu-Lie algebras, introduced in [6] by Ataguema, Makhlouf and Silvestrov. They correspond to a generalized version by twisting of n-ary Nambu algebras and Nambu-Lie algebras which are called Filippov algebras. We deal in this paper with a subclass of n-ary Hom-Nambu algebras called multiplicative n-ary Hom-Nambu algebras.

Definition 1.1. An *n-ary Hom-Nambu* algebra is a triple $(\mathcal{N},[\cdot,...,\cdot],\widetilde{\alpha})$ consisting of a vector space $\mathcal{N}$, an n-linear map $[\cdot,...,\cdot]:\mathcal{N}^n\longrightarrow\mathcal{N}$ and a family $\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ of linear maps $\alpha_i:\ \ \mathcal{N}\longrightarrow\mathcal{N}$, satisfying

$$\begin{aligned}&\big[\alpha_1(x_1),....,\alpha_{n-1}(x_{n-1}),[y_1,....,y_n]\big]=\\&\sum_{i=1}^{n}\big[\alpha_1(y_1),....,\alpha_{i-1}(y_{i-1}),[x_1,....,x_{n-1},y_i],\alpha_i(y_{i+1}),...,\alpha_{n-1}(y_n)\big],\end{aligned}\tag{1.1}$$

for all $(x_1,...,x_{n-1})\in\mathcal{N}^{n-1}$, $(y_1,...,y_n)\in\mathcal{N}^n$.
The identity (1.1) is called *Hom-Nambu identity*.

Let $x=(x_1,\dots,x_{n-1})\in\mathcal{N}^{n-1}$, $\widetilde{\alpha}(x)=(\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}))\in\mathcal{N}^{n-1}$ and $y\in\mathcal{N}$. We define an adjoint map $ad(x)$ as a linear map on $\mathcal{N}$, such that

$$ad(x)(y)=[x_1,\cdots,x_{n-1},y].\tag{1.2}$$

Then the Hom-Nambu identity (1.1) may be written in terms of adjoint map as

$$ad(\widetilde{\alpha}(x))([x_n,...,x_{2n-1}])=\sum_{i=n}^{2n-1}[\alpha_1(x_n),...,\alpha_{i-n}(x_{i-1}),ad(x)(x_i),\alpha_{i-n+1}(x_{i+1})...,\alpha_{n-1}(x_{2n-1})].$$

Remark 1.2. When the maps $(\alpha_i)_{1\leq i\leq n-1}$ are all identity maps, one recovers the classical n-ary Nambu algebras. The Hom-Nambu Identity (1.1), for $n=2$, corresponds to Hom-Jacobi identity (see [20]), which reduces to Jacobi identity when $\alpha_1=id$.

Let $(\mathcal{N},[\cdot,\dots,\cdot],\widetilde{\alpha})$ and $(\mathcal{N}',[\cdot,\dots,\cdot]',\widetilde{\alpha}')$ be two n-ary Hom-Nambu algebras where $\widetilde{\alpha}=(\alpha_i)_{i=1,\cdots,n-1}$ and $\widetilde{\alpha}'=(\alpha'_i)_{i=1,\cdots,n-1}$. A linear map $f:\mathcal{N}\to\mathcal{N}'$ is an n-ary Hom-Nambu algebras *morphism* if it satisfies

$$\begin{aligned}f([x_1,\cdots,x_n])&=[f(x_1),\cdots,f(x_n)]'\\f\circ\alpha_i&=\alpha'_i\circ f\quad\forall i=1,n-1.\end{aligned}$$

Definition 1.3. An n-ary Hom-Nambu algebra $(\mathcal{N},[\cdot,...,\cdot],\widetilde{\alpha})$ where $\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ is called *n-ary Hom-Nambu-Lie* algebra if the bracket is skew-symmetric that is

$$[x_{\sigma(1)},..,x_{\sigma(n)}]=Sgn(\sigma)[x_1,..,x_n],\quad\forall\sigma\in\mathcal{S}_n\ \text{ and }\ \forall x_1,...,x_n\in\mathcal{N}.\tag{1.3}$$

where $\mathcal{S}_n$ stands for the permutation group of n elements.

In the sequel we deal with a particular class of n-ary Hom-Nambu-Lie algebras which we call n-ary multiplicative Hom-Nambu-Lie algebras.

Definition 1.4. An *n-ary multiplicative Hom-Nambu algebra* (resp. *n-ary multiplicative Hom-Nambu-Lie algebra*) is an n-ary Hom-Nambu algebra (resp. n-ary Hom-Nambu-Lie algebra) $(\mathcal{N},[\cdot,...,\cdot],\widetilde{\alpha})$ with $\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ where $\alpha_1=...=\alpha_{n-1}=\alpha$ and satisfying

$$\alpha([x_1,..,x_n])=[\alpha(x_1),..,\alpha(x_n)],\quad\forall x_1,...,x_n\in\mathcal{N}.\tag{1.4}$$

For simplicity, we will denote the n-ary multiplicative Hom-Nambu algebra as $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ where $\alpha:\mathcal{N}\to\mathcal{N}$ is a linear map. Also by misuse of language an element $x\in\mathcal{N}^n$ refers $x=(x_1,..,x_n)$ and $\alpha(x)$ denotes $(\alpha(x_1),...,\alpha(x_n))$.

The following theorem gives a way to construct n-ary multiplicative Hom-Nambu algebras (resp. Hom-Nambu-Lie algebras) starting from classical n-ary Nambu algebras (resp. Nambu-Lie algebras) and algebra endomorphisms.

Theorem 1.5. [6] *Let $(\mathcal{N},[\cdot,...,\cdot])$ be an n-ary Nambu algebra (resp. n-ary Nambu-Lie algebra) and let $\rho:\mathcal{N}\to\mathcal{N}$ be an n-ary Nambu (resp. Nambu-Lie) algebra endomorphism. Then $(\mathcal{N},\rho\circ[\cdot,...,\cdot],\rho)$ is a n-ary multiplicative Hom-Nambu algebra (resp. n-ary multiplicative Hom-Nambu-Lie algebra).*

2. Representations of Hom-Nambu-Lie algebras

In this section we extend the representation theory of Hom-Lie algebras introduced in [30] and [10] to the n-ary case. We denote by $End(\mathcal{N})$ the linear group of operators on the $\mathbb{K}$-vector space $\mathcal{N}$. Sometimes it is considered as a Lie algebra with the commutator brackets.

2.1. Derivations of n-ary Hom-Nambu-Lie algebras.

Let $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ be an n-ary multiplicative Hom-Nambu-Lie algebra. We denote by α^k the k-times composition of α (i.e. $\alpha^k=\alpha\circ...\circ\alpha$ k-times). In particular $\alpha^{-1}=0$ and $\alpha^0=id$.

Definition 2.1. For any $k\geq 1$, we call $D\in End(\mathcal{N})$ an *α^k-derivation* of the n-ary multiplicative Hom-Nambu-Lie algebra $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ if

$$[D,\alpha]=0 \quad (\text{i.e. } D\circ\alpha=\alpha\circ D), \tag{2.1}$$

and

$$D[x_1,...,x_n]=\sum_{i=1}^{n}[\alpha^k(x_1),...,\alpha^k(x_{i-1}),D(x_i),\alpha^k(x_{i+1}),...,\alpha^k(x_n)], \tag{2.2}$$

We denote by $Der_{\alpha^k}(\mathcal{N})$ the set of α^k-derivations of the n-ary multiplicative Hom-Nambu-Lie algebra $\mathcal{N}$.

For $x=(x_1,...,x_{n-1})\in\mathcal{N}^{n-1}$ satisfying $\alpha(x)=x$ and $k\geq 1$, we define the map $ad_k(x)\in End(\mathcal{N})$ by

$$ad_k(x)(y)=[x_1,...,x_{n-1},\alpha^k(y)] \quad \forall y\in\mathcal{N}. \tag{2.3}$$

Then

Lemma 2.2. *The map $ad_k(x)$ is an α^{k+1}-derivation, that we call inner α^{k+1}-derivation.*

We denote by $Inn_{\alpha^k}(\mathcal{N})$ the $\mathbb{K}$-vector space generated by all inner α^{k+1}-derivations. For any $D\in Der_{\alpha^k}(\mathcal{N})$ and $D'\in Der_{\alpha^k}(\mathcal{N})$ we define their commutator $[D,D']$ as usual:

$$[D,D']=D\circ D'-D'\circ D. \tag{2.4}$$

Set $Der(\mathcal{N})=\bigoplus_{k\geq -1}Der_{\alpha^k}(\mathcal{N})$ and $Inn(\mathcal{N})=\bigoplus_{k\geq -1}Inn_{\alpha^k}(\mathcal{N})$.

Lemma 2.3. *For any $D\in Der_{\alpha^k}(\mathcal{N})$ and $D'\in Der_{\alpha^{k'}}(\mathcal{N})$, where $k+k'\geq -1$, we have*

$$[D,D']\in Der_{\alpha^{k+k'}}(\mathcal{N}).$$

Proof. Let $x_i \in \mathcal{N},\ 1 \leq i \leq n$, $D \in Der_{\alpha^k}(\mathcal{N})$ and $D' \in Der_{\alpha^{k'}}(\mathcal{N})$, then

$$\begin{aligned}
D \circ D'([x_1, ..., x_n]) &= \sum_{i=1}^{n} D([\alpha^{k'}(x_1), ..., D'(x_i), ..., \alpha^{k'}(x_n)]) \\
&= \sum_{i=1}^{n} [\alpha^{k+k'}(x_1), ..., D \circ D'(x_i), ..., \alpha^{k+k'}(x_n)] \\
&+ \sum_{i<j}^{n} [\alpha^{k+k'}(x_1), ..., \alpha^{k}(D'(x_i)), ...\alpha^{k'}(D(x_j)), ..., \alpha^{k+k'}(x_n)] \\
&+ \sum_{i>j}^{n} [\alpha^{k+k'}(x_1), ..., \alpha^{k'}(D(x_j)), ...\alpha^{k}(D'(x_i)), ..., \alpha^{k+k'}(x_n)].
\end{aligned}$$

The second and the third term in $[D, D']$ are symmetrical, then

$$\begin{aligned}
[D, D']([x_1, ..., x_n]) &= (D \circ D' - D' \circ D)([x_1, ..., x_n]) \\
&= \sum_{i=1}^{n} [\alpha^{k+k'}(x_1), ..., (D \circ D' - D' \circ D)(x_i), ..., \alpha^{k+k'}(x_n)] \\
&= \sum_{i=1}^{n} [\alpha^{k+k'}(x_1), ..., [D, D'](x_i), ..., \alpha^{k+k'}(x_n)],
\end{aligned}$$

which yield that $[D, D'] \in Der_{\alpha^{k+k'}}(\mathcal{N})$. $\square$

Moreover we have:

Proposition 2.4. *The pair $(Der(\mathcal{N}), [\cdot, \cdot])$, where the bracket is the usual commutator, defines a Lie algebra and $Inn(V)$ constitutes an ideal of it.*

Proof. $(Der(\mathcal{N}), [\cdot, \cdot])$ is a Lie algebra by using Lemma 2.3. We show that $Inn(V)$ is an ideal. Let $ad_k(x) = [x_1, ..., x_{n-1}, \alpha^{k-1}(\cdot)]$ be an inner α^k-derivation on $\mathcal{N}$ and $D \in Der_{\alpha^{k'}}(\mathcal{N})$ for $k \geq -1$ and $k' \geq -1$ where $k + k' \geq -1$. Then

$$[D, ad_k(x)] \in Der_{\alpha^{k+k'}}(\mathcal{N})$$

and for any $y \in \mathcal{N}$

$$\begin{aligned}
[D, ad_k(x)](y) &= D([x_1, ..., x_{n-1}, \alpha^{k-1}(y)]) - [x_1, ..., x_{n-1}, \alpha^{k-1}(D(y))], \\
&= D([\alpha^{k}(x_1), ..., \alpha^{k}(x_{n-1}), \alpha^{k-1}(y)]) - [\alpha^{k+k'}(x_1), ..., \alpha^{k+k'}(x_{n-1}), \alpha^{k-1}(D(y))], \\
&= \sum_{i \leq n-1} [\alpha^{k+k'}(x_1), ..., D(\alpha^{k}(x_i)), ..., \alpha^{k+k'}(x_{n-1}), \alpha^{k+k'-1}(y)], \\
&= \sum_{i \leq n-1} [x_1, ..., D(x_i), ..., x_{n-1}, \alpha^{k+k'-1}(y)], \\
&= \sum_{i \leq n-1} ad_{k+k'}(x_1 \wedge ... \wedge D(x_i) \wedge ... \wedge x_{n-1})(y).
\end{aligned}$$

Therefore $[D, ad_k(x)] \in Inn_{\alpha^{k+k'}}(V)$. $\square$

2.2. **Representations of n-ary Hom-Nambu-Lie algebras.** In this section we introduce and study the representations of n-ary multiplicative Hom-Nambu-Lie algebras.

Definition 2.5. A representation of an n-ary multiplicative Hom-Nambu-Lie algebra $(\mathcal{N}, [\cdot, ..., \cdot], \alpha)$ on a vector space $\mathcal{N}$ is a skew-symmetric multilinear map $\rho : \mathcal{N}^{n-1} \longrightarrow End(\mathcal{N})$, satisfying for $x, y \in \mathcal{N}^{n-1}$ the identity

$$\rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x) = \sum_{i=1}^{n-1} \rho(\alpha(x_1), ..., ad(y)(x_i), ..., \alpha(x_{n-1})) \circ \nu \tag{2.5}$$

where ν is a linear map.

Two representations ρ and ρ' on $\mathcal{N}$ are *equivalent* if there exists $f:\mathcal{N}\to\mathcal{N}$ an isomorphism of vector space such that $f(x\cdot y)=x\cdot' f(y)$ where $x\cdot y=\rho(x)(y)$ and $x\cdot' y=\rho'(x)(y)$ for $x\in\mathcal{N}^{n-1}$ and $y\in\mathcal{N}$.

Example 2.6. *Let $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ be an n-ary multiplicative Hom-Nambu-Lie algebra. The map* ad *defined in* (1.2) *is a representation, where the operator ν is the twist map α. The identity* (2.5) *is equivalent to Hom-Nambu identity. It is called the adjoint representation.*

3. From n-ary Hom-Nambu-Lie algebra to Hom-Leibniz algebra

In the context of Hom-Lie algebras one gets the class of Hom-Leibniz algebras (see [20]). Following the standard Loday's conventions for Leibniz algebras, a Hom-Leibniz algebra is a triple $(V,[\cdot,\cdot],\alpha)$ consisting of a vector space V, a bilinear map $[\cdot,\cdot]:V\times V\to V$ and a linear map $\alpha:V\to V$ with respect to $[\cdot,\cdot]$ satisfying

$$[\alpha(x),[y,z]]=[[x,y],\alpha(z)]+[\alpha(y),[x,z]] \tag{3.1}$$

Let $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ be a n-ary multiplicative Hom-Nambu-Lie algebras, we define

• a linear map $L:\wedge^{n-1}\mathcal{N}\longrightarrow End(\mathcal{N})$ by

$$L(x)\cdot z=[x_1,...,x_{n-1},z], \tag{3.2}$$

for all $x=x_1\wedge...\wedge x_{n-1}\in\wedge^{n-1}\mathcal{N},\ z\in\mathcal{N}$ and extending it linearly to all $\wedge^{n-1}\mathcal{N}$. Notice that $L(x)\cdot z=ad(x)(z)$.

• a linear map $\tilde{\alpha}:\wedge^{n-1}\mathcal{N}\longrightarrow\wedge^{n-1}\mathcal{N}$ by

$$\tilde{\alpha}(x)=\alpha(x_1)\wedge...\wedge\alpha(x_{n-1}) \tag{3.3}$$

for all $x=x_1\wedge...\wedge x_{n-1}\in\wedge^{n-1}\mathcal{N}$,

• a bilinear map $[\ ,\ ]_\alpha:\wedge^{n-1}\mathcal{N}\times\wedge^{n-1}\mathcal{N}\longrightarrow\wedge^{n-1}\mathcal{N}$ by

$$[x,y]_\alpha=L(x)\bullet_\alpha y=\sum_{i=0}^{n-1}\big(\alpha(y_1),...,L(x)\cdot y_i,...,\alpha(y_{n-1})\big), \tag{3.4}$$

for all $x=x_1\wedge...\wedge x_{n-1}\in\wedge^{n-1}\mathcal{N},\ y=y_1\wedge...\wedge y_{n-1}\in\wedge^{n-1}\mathcal{N}$

We denote by $\mathcal{L}(\mathcal{N})$ the space $\wedge^{n-1}\mathcal{N}$ and we call it the fundamental set.

Lemma 3.1. *The map L satisfies*

$$L([x,y]_\alpha)\cdot\alpha(z)=L(\alpha(x))\cdot\big(L(y)\cdot z\big)-L(\alpha(y))\cdot\big(L(x)\cdot z\big) \tag{3.5}$$

for all $x,\ y\in\mathcal{L}(\mathcal{N}),\ z\in\mathcal{N}$

Proposition 3.2. *The triple $\big(\mathcal{L}(\mathcal{N}),\ [\ ,\ ]_\alpha,\ \alpha\big)$ is a Hom-Leibniz algebra.*

Proof. Let $x=x_1\wedge...\wedge x_{n-1},\ y=y_1\wedge...\wedge y_{n-1}$ and $z=z_1\wedge...\wedge z_{n-1}\in\mathcal{L}(\mathcal{N})$, the Leibniz identity (3.1) can be written

$$\big[[x,y]_\alpha,\alpha(z)\big]_\alpha=\big[\alpha(x),[y,z]_\alpha\big]_\alpha-\big[\alpha(y),[x,z]_\alpha\big]_\alpha \tag{3.6}$$

and equivalently

$$\Big(L\big(L(x)\bullet_\alpha y\big)\bullet_\alpha\tilde{\alpha}(z)\Big)\cdot(v)=\Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha z\big)\Big)\cdot(v)-\Big(L(\alpha(y))\bullet_\alpha\big(L(x)\bullet_\alpha z\big)\Big)\cdot(v). \tag{3.7}$$

Let us compute first $\Big(L(\tilde{\alpha}(x))\bullet_\alpha\big(L(y)\bullet_\alpha z\big)\Big)$. This is given by

$$\begin{aligned}\Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha z\big)\Big) &= \sum_{i=0}^{n-1} L(\alpha(x))\bullet_\alpha\big(\alpha(z_1),...,L(y)\cdot z_i,...,\alpha(z_{n-1})\big)\\ &= \sum_{i=0}^{n-1}\sum_{i\neq j,j=0}^{n-1}\big(\alpha^2(z_1),...,\alpha(L(x)\cdot z_j),...,\alpha(L(y)\cdot z_i)...,\alpha^2(z_{n-1})\big)\\ &+ \sum_{i=0}^{n-1}\big(\alpha^2(z_1),...,L(\tilde{\alpha}(x))\cdot(L(y)\cdot z_i),...,\alpha^2(z_{n-1})\big).\end{aligned}$$

The right hand side of (3.7) is skewsymmetric in x, y; hence,

$$\Big(L(\alpha(x))\bullet_\alpha\big(L(y)\bullet_\alpha z\big)\Big)-\Big(L(\alpha(y))\bullet_\alpha\big(L(x)\bullet_\alpha z\big)\Big)=$$

$$\sum_{i=0}^{n-1}(\alpha^2(z_1),...,\{L(\alpha(x))\cdot(L(y)\cdot z_i)-L(\alpha(y))\cdot(L(x).z_i)\},...,\alpha^2(z_{n-1})\big). \tag{3.8}$$

In the other hand, using Definition (3.4), we find

$$\Big(L\big(L(x)\bullet_\alpha y\big)\bullet_\alpha\tilde{\alpha}(z)\Big)=$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\big(\alpha^2(z_1),...,\alpha^2(z_{i-1}),[\alpha(y_1),...,L(x)\cdot y_j,...,\alpha(y_{n-1}),\alpha(z_i)],\alpha^2(z_{i+1}),...,\alpha^2(z_{n-1})\big)$$

$$=\sum_{i=0}^{n-1}\big(\alpha^2(z_1),...,\alpha^2(z_{i-1}),[x,y]_\alpha\cdot\alpha(z_i),\alpha^2(z_{i+1}),...,\alpha^2(z_{n-1})\big). \tag{3.9}$$

Using Lemma 3.1, the proof is completed. □

Remark 3.3. We obtain a similar result if we consider the space $T\mathcal{N}=\otimes^n\mathcal{N}$ instead of $\mathcal{L}(\mathcal{N})$.

Remark 3.4. For $n=2$ the map $L:\mathcal{L}(\mathcal{N})\longrightarrow End(\mathcal{N})$ defines a representation of $\mathcal{L}(\mathcal{N})$ on $\mathcal{N}$. One should set $\nu=\alpha$ and check

$$L(\alpha(x))\cdot\alpha(z)=\alpha(L(x)\cdot z) \tag{3.10}$$

$$L([x,y]_\alpha)\cdot\alpha(z)=L(\alpha(x))(y)\cdot z-L(\alpha(y))(x)\cdot z \tag{3.11}$$

Indeed (3.10) and (3.11) are equivalent to

$$[\alpha(x),\alpha(y)]=\alpha([x,y]), \tag{3.12}$$

$$[[x,y],\alpha(z)]=[[\alpha(x),y],z]-[[\alpha(y),x],z]. \tag{3.13}$$

According to [30] and [10] it corresponds to the adjoint representation of a Hom-Lie algebra.

4. Central Extensions and Cohomology of n-ary Hom-Nambu-Lie algebras

4.1. Central extensions of n-ary multiplicative Hom-Nambu-Lie algebras.

Let $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ be an n-ary multiplicative Hom-Nambu-Lie algebra.

Definition 4.1. We define a central extension $\tilde{\mathcal{N}}$ of $\mathcal{N}$ by adding a new central generator e and modifying the bracket as follows: for all $\tilde{x}_i=x_i+a_ie$, $a_i\in\mathbb{K}$ and $1\leq i\leq n$ we have

$$[\tilde{x}_1,...,\tilde{x}_n]_{\widetilde{\mathcal{N}}}=[x_1,...,x_n]+\varphi(x_1,...,x_n)e, \tag{4.1}$$

$$\beta(\widetilde{x}_i)=\alpha(x_i)+\lambda(x_i)e, \tag{4.2}$$

$$[\tilde{x}_1,...,\tilde{x}_{n-1},e]_{\widetilde{\mathcal{N}}}=0, \tag{4.3}$$

where $\lambda:\mathcal{N}\to\mathbb{K}$ a linear map.

One may think of adding more than one central generator, but this will not be needed here for the discussion.

- Clearly, φ has to be an n-linear and skew-symmetric map, $\varphi \in \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$, where $\mathcal{N}^*$ is the dual of $\mathcal{N}$. It will be identified with a 1-cochain.
- The new bracket for the $\tilde{x}_i \in \widetilde{\mathcal{N}}$ has to satisfy the Hom-Nambu identity. This leads to a condition on φ when one of the vector involved is e.
- Since e is a central then the he Hom-Nambu identity has no restriction on λ.
  For $\tilde{x}_i = x_i + a_i e \in \tilde{\mathcal{N}}$, $\tilde{y}_i = y_i + b_i e \in \tilde{\mathcal{N}}$, $1 \leq i \leq n$, we have

$$\begin{aligned}&\big[\beta(\tilde{x}_1), ..., \beta(\tilde{x}_{n-1}), [\tilde{y}_1, ..., \tilde{y}_n]_{\widetilde{\mathcal{N}}}\big]_{\widetilde{\mathcal{N}}} = \\ &\sum_{i=1}^{n-1} \big[\beta(\tilde{y}_1), ..., \beta(\tilde{y}_{i-1}), [\tilde{x}_1, ..., \tilde{x}_{n-1}, y_i]_{\widetilde{\mathcal{N}}}.\beta(\tilde{y}_{i+1}), ..., \beta(\tilde{y}_n)\big]_{\widetilde{\mathcal{N}}},\end{aligned}$$

Using (4.1) and the Hom-Nambu identity for the original Hom-Nambu-Lie algebra, one gets

$$\begin{aligned}&\varphi\big(\alpha(x_1), ..., \alpha(x_{n-1}), [y_1, ..., y_n]\big) - \\ &\sum_{i=1}^{n-1} \varphi\big(\alpha(y_1), ..., \alpha(y_{i-1}), [x_1, ..., x_{n-1}, y_i], \alpha(y_{i+1}), ..., \alpha(y_n)\big) = 0,\end{aligned} \tag{4.4}$$

- The previous equation, may be written as

$$\delta^2\varphi(x, y, z) = 0$$

  where $x = x_1 \otimes ... \otimes x_{n-1} \in \mathcal{N}^{\otimes n-1}$, $y = y_1 \otimes ... \otimes y_{n-1} \in \mathcal{N}^{\otimes n-1}$, $z = y_n \in \mathcal{N}$.
  We provide below the condition that characterizes $\varphi \in \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$, $\varphi : x \wedge z \to \varphi(x, z)$ as a 1-cocycle. It is seen now why becomes natural to call φ a 1-cocycle (rather than a 2-cochain, as it is in the Hom-Lie cohomology case in [22]).
  The number of elements of $\mathcal{L}(\mathcal{N})$ in the argument of a cochain determines its order. As we shall see shortly, an arbitrary p-cochain takes $p(n-1)+1$ arguments in $\mathcal{N}$. A 0-cochain is an element of $\mathcal{N}^*$.

4.2. **Cohomology adapted to central extensions of multiplicative Hom-Nambu-Lie algebras.** Let us now construct the cohomology complex relevant for central extensions of multiplicative Hom-Nambu-Lie algebras. Since $\mathcal{N}$ does not act on $\varphi(x, z)$, it will be the cohomology of multiplicative Hom-Nambu-Lie algebras for the trivial action.

Definition 4.2. We define an arbitrary p-cochain as an element $\varphi \in \wedge^{n-1}\mathcal{N}^* \otimes ... \otimes \wedge^{n-1}\mathcal{N}^* \wedge \mathcal{N}^*$,

$$\begin{array}{rcl}\varphi : \mathcal{L}(\mathcal{N}) \otimes ... \otimes \mathcal{L}(\mathcal{N}) \wedge \mathcal{N} & \longrightarrow & \mathbb{K} \\ (x_1, .., x_p, z) & \longmapsto & \varphi(x_1, .., x_p, z)\end{array}$$

We denote the set of p-cochains with values in $\mathbb{K}$ by $C^p(\mathcal{N}, \mathbb{K})$.

Condition (4.4) guarantees the consistency of φ according to (4.1) with the Hom-Nambu identity (1.1). Then

$$\delta^2\varphi(x, y, z) = \varphi\big(\alpha(x), L(y) \cdot z\big) - \varphi\big(\alpha(y), L(x) \cdot z\big) - \varphi\big([x, y]_\alpha, \alpha(z)\big) = 0, \tag{4.5}$$

where $L(x) \cdot z$ and $[x, y]_\alpha$ are defined in (3.2) and (3.4). It is now straightforward to extend (4.5) to a whole cohomology complex; $\delta^p\varphi$ will be a $(p+1)$-cochain taking one more argument of $\mathcal{L}(\mathcal{N})$ than φ. This is done by means of the following

Definition 4.3. Let $\varphi \in C^p(\mathcal{N}, \mathbb{K})$ be a p-cochain on a multiplicative n-ary Hom-Nambu-Lie algebra $\mathcal{N}$. A coboundary operator δ^p on arbitrary p-cochain is given by

$$\begin{aligned}(4.6)\qquad \delta^p\varphi(x_1,...,x_{p+1},z) &= \sum_{1\leq i<j}^{p+1}(-1)^i\varphi\big(\alpha(x_1),...,\hat{x}_i,...,[x_i,x_j]_\alpha,...,\alpha(x_{p+1}),\alpha(z)\big)\\ &+ \sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(x_1),...,\hat{x}_i,...,\alpha(x_{p+1}),L(x_i)\cdot z\big)\end{aligned}$$

where $x_1,...,x_{p+1}\in\mathcal{L}(\mathcal{N}),\ z\in\mathcal{N}$ and $\hat{x}_i$ designed that x_i is omitted.

Proposition 4.4. *If $\varphi\in C^p(\mathcal{N},\mathbb{K})$ be a p-cochain, then*

$$\delta^{p+1}\circ\delta^p(\varphi)=0$$

Proof. Let φ be a p-cochain, $(x_i)_{1\leq i\leq p}\in\mathcal{L}(\mathcal{N})$ et $z\in\mathcal{N}$, we can write δ^p and $\delta^{p+1}\circ\delta^p$ as

$$\begin{aligned}&\delta^p=\delta_1^p+\delta_2^p\\ \text{and}\qquad &\delta^{p+1}\circ\delta^p=\eta_{11}+\eta_{12}+\eta_{21}+\eta_{22}\end{aligned}$$

where $\eta_{ij}=\delta_i^{p+1}\circ\delta_j^p$, $1\leq i,j\leq 2$, and

$$\begin{aligned}\delta_1^p\varphi(x_1,...,x_{p+1},z) &= \sum_{1\leq i<j}^{p+1}(-1)^i\varphi\big(\alpha(x_1),...,\hat{x}_i,...,[x_i,x_j]_\alpha,...,\alpha(x_{p+1}),\alpha(z)\big)\\ \delta_2^p\varphi(x_1,...,x_{p+1},z) &= \sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(x_1),...,\hat{x}_i,...,\alpha(x_{p+1}),L(x_i)\cdot z\big)\end{aligned}$$

• Let us compute first $\eta_{11}\varphi(x_1,...,x_{p+1},z)$. This is given by

$$\begin{aligned}&\eta_{11}(\varphi)(x_1,...,x_{p+1},z)\\ =&\sum_{1\leq i<k<j}^{p+1}(-1)^{i+k}\varphi\big(\alpha^2(x_1),...,\widehat{x}_i,...,\widehat{\alpha(x_k)},...,[\alpha(x_k),[x_i,x_j]_\alpha]_\alpha,....,\alpha^2(x_{p+1}),\alpha^2(z)\big)\\ +&\sum_{1\leq i<k<j}^{p+1}(-1)^{i+k-1}\varphi\big(\alpha^2(x_1),...,\widehat{\alpha(x_i)},...,\widehat{x_k},...,[\alpha(x_i),[x_k,x_j]_\alpha]_\alpha,....,\alpha^2(x_{p+1}),\alpha^2(z)\big)\\ +&\sum_{1\leq i<k<j}^{p+1}(-1)^{i+k-1}\varphi\big(\alpha^2(x_1),...,\widehat{x}_i,...,\widehat{[x_i,x_k]}_\alpha,...,[[x_i,x_k]_\alpha,\alpha(x_j)]_\alpha,....,\alpha^2(x_{p+1}),\alpha^2(z)\big).\end{aligned}$$

Whence applying the Hom-Leibniz identity (3.6) to $x_i,\ x_j,\ x_k\in\mathcal{L}(\mathcal{N})$, we find $\eta_{11}=0$.

•

$$\begin{aligned}&\eta_{21}(\varphi)(x_1,...,x_{p+1},z)+\eta_{12}(\varphi)(x_1,...,x_{p+1},z)=\\ &\sum_{1\leq i<j}^{p+1}(-1)^{i-1}\varphi\big(\alpha^2(x_1),...,\widehat{x}_i,...,\widehat{[x_i,x_j]}_\alpha,...,\alpha^2(x_{p+1}),L([x_i,x_j]_\alpha)\cdot\alpha(z)\big)\end{aligned}$$

and

$$\begin{aligned}&\eta_{22}(\varphi)(x_1,...,x_{p+1},z)\\ =&\sum_{1\leq i<j}^{p+1}(-1)^i\varphi\big(\alpha^2(x_1),...,\widehat{x}_i,...,\widehat{\alpha(x_j)},...,\alpha^2(x_{p+1}),\big(L(\alpha(x_i))\cdot(L(x_j)\cdot z)\big)\big)\\ +&\sum_{1\leq i<j}^{p+1}(-1)^{i-1}\varphi\big(\alpha^2(x_1),...,\widehat{\alpha(x_i)},...,\widehat{x}_j,...,\alpha^2(x_{p+1}),\big(L(\alpha(x_j))\cdot(L(x_i)\cdot z)\big)\big).\end{aligned}$$

Then applying the Lemma 3.1 to $x_i,\ x_j \in \mathcal{L}(\mathcal{N})$ and $z \in \mathcal{N}$, $\eta_{12}+\eta_{21}+\eta_{22}=0$.
Which ends the proof $\square$

Definition 4.5. The space of p-cocycles is defined by

$$Z^p(\mathcal{N},\mathbb{K}) = \{\varphi \in C^p(\mathcal{N},\mathbb{K}) : \delta^p\varphi = 0\}$$

and the space of p-coboundaries is defined by

$$B^p(\mathcal{N},\mathbb{K}) = \{\psi = \delta^{p-1}\varphi : \varphi \in C^{p-1}(\mathcal{N},\mathbb{K})\}$$

Lemma 4.6. $B^p(\mathcal{N},\mathbb{K}) \subset Z^p(\mathcal{N},\mathbb{K})$

Definition 4.7. We call p^{th}-cohomology group the quotient

$$H^p(\mathcal{N},\mathbb{K}) = \frac{Z^p(\mathcal{N},\mathbb{K})}{B^p(\mathcal{N},\mathbb{K})}$$

Example 4.8. *Let* $(\mathcal{N},[\cdot,...,\cdot])$ *be a Nambu-Lie algebra (see* [14] [17]*) and* $\{e_i\}_{i=1}^{n+1}$ *be a basis such that*

$$[e_1,...,\hat{e}_i,...,e_{n+1}] = (-1)^{i+1}\varepsilon_i e_i \quad \textit{or} \quad [e_{i_1},....,e_{i_n}] = (-1)^n \sum_{i=1}^{n+1} \varepsilon_i \epsilon^i_{i_1,...,i_n} e_i \tag{4.7}$$

where $\varepsilon_i = \pm 1$ *(no sum over the* i *of the* ε_i *factors) just introduce signs that affect the different terms of the sum in* i *and we have used Filippov's notation.*

Note that we might equally well have the $\epsilon^i_{i_1,...,i_n}$ *without signs* ε_i *in 4.7 by taking* $\epsilon^i_{i_1,...,i_n} = \eta^{ij}\epsilon_{i_1,...,i_n,j}$*, where* $\epsilon_{1,...,n,(n+1)} = 1$ *and* η *is a* $(n+1)\times(n+1)$ *diagonal matrix with* $+1$ *and* -1 *in places indicated by the* ε_i*'s. We shall keep nevertheless the customary* ε_i *factors above as in e.g.* [17].

Let $\alpha : \mathcal{N} \to \mathcal{N}$ *be a morphism of Nambu-Lie algebras. Then using Theorem 1.5,* $\mathcal{N}_\alpha = (\mathcal{N},[\cdot,...,\cdot]_\alpha,\tilde{\alpha} = (\alpha,...,\alpha))$ *is a Hom-Nambu-Lie algebra where the bracket* $[\cdot,...,\cdot]_\alpha$ *is given by*

$$[e_1,...,\hat{e}_i,...,e_{n+1}]_\alpha = (-1)^{i+1}\varepsilon_i\alpha(e_i) \quad \textit{or} \quad [e_{i_1},....,e_{i_n}]_\alpha = (-1)^n \sum_{i=1}^{n+1} \varepsilon_i \epsilon^i_{i_1,...,i_n}\alpha(e_i). \tag{4.8}$$

We establish the following result.

Lemma 4.9. *Any* 1*-cochain of the Hom-Nambu-Lie algebra* $\mathcal{N}_\alpha$ *is a* 1*-coboundary (and thus a trivial* 1*-cocycle).*

Proof. Let $\varphi \in C^1(\mathcal{N},\mathbb{K})$ be a 1-cochain on $\mathcal{N}_\alpha$, φ is determined by its coordinates $\varphi_{i_1,...,i_n} = \varphi(e_{i_1},...,e_{i_n})$. We now show that, in fact, a 1-cochain on $\mathcal{N}_\alpha$ is a 1-coboudary, that is there exists a 0-cochain ϕ such that

$$\varphi_{i_1,...,i_n} = -\phi([e_{i_1},...,e_{i_n}]) = -\sum_{k=1}^{n+1}\varepsilon_k\epsilon^k_{i_1,...,i_n}\phi_k, \tag{4.9}$$

where $\phi_k = \phi\circ\alpha(e_k)$. Indeed, given φ then the 0-cochain ϕ is given by

$$\phi_k = -\frac{\varepsilon_k}{n!}\sum_{i_1...i_n}^{n+1}\epsilon_k^{i_1,...,i_n}\varphi_{i_1,...,i_n} \tag{4.10}$$

has the desired property (4.9):

$$
\begin{aligned}
-\phi([e_{i_1},...,e_{i_n}]) &= -\sum_{k=1}^{n+1}\varepsilon_k\epsilon^k_{i_1,\dots,i_n}\phi_k \\
&= \sum_{k=1}^{n+1}\epsilon^k_{i_1,\dots,i_n}\frac{\varepsilon_k^2}{n!}\sum_{j_1\cdots j_n}^{n+1}\epsilon_k^{j_1,\dots,j_n}\varphi_{j_1,\dots,j_n} \\
&= \frac{1}{n!}\sum_{j_1\cdots j_n}^{n+1}\epsilon^{j_1,\dots,j_n}_{i_1,\dots,i_n}\varphi_{j_1,\dots,j_n} = \varphi_{i_1,\dots,i_n}
\end{aligned}
\tag{4.11}
$$

which proves the lemma. □

5. Deformation of n-ary Hom-Nambu-Lie algebras

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $\mathcal{N}[[t]]$ be the set of formal series whose coefficients are elements of the vector space $\mathcal{N}$, ($\mathcal{N}[[t]]$ is obtained by extending the coefficients domain of $\mathcal{N}$ from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Given a $\mathbb{K}$-n-linear map $\varphi : \mathcal{N}\times ... \times\mathcal{N}\to\mathcal{N}$, it admits naturally an extension to a $\mathbb{K}[[t]]$-n-linear map $\varphi : \mathcal{N}[[t]]\times ... \times\mathcal{N}[[t]]\to\mathcal{N}[[t]]$, that is, if $x_i = \sum_{j\geq 0} a_i^j t^j$, $1\leq i\leq n$ then $\varphi(x_1,...,x_n) = \sum_{j_1,\dots,j_n\geq 0} t^{j_1+\dots+j_n}\varphi(a_1^{j_1},...,a_n^{j_n})$. The same holds for linear map.

Definition 5.1. Let $(\mathcal{N},[\cdot,...,\cdot],\widetilde{\alpha})$, $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ be a Hom-Nambu-Lie algebra. A formal deformation of the Hom-Nambu-Lie algebra $\mathcal{N}$ is given by a $\mathbb{K}[[t]]$-n-linear map

$$[\cdot,...,\cdot]_t : \mathcal{N}[[t]]\times ... \times\mathcal{N}[[t]]\to\mathcal{N}[[t]]$$

of the form $[\cdot,...,\cdot]_t = \sum_{i\geq 0} t^i[\cdot,...,\cdot]_i$ where each $[\cdot,...,\cdot]_i$ is a $\mathbb{K}[[t]]$-n-linear map $[\cdot,...,\cdot]_i : \mathcal{N}\times ...\times\mathcal{N}\to\mathcal{N}$ (extending to be $\mathbb{K}[[t]]$-n-linear), and $[\cdot,...,\cdot]_0 = [\cdot,...,\cdot]$ such that for $(x_i)_{1\leq i\leq n-1}$, $(y_i)_{1\leq i\leq n}\in\mathcal{N}$

$$\big[\alpha_1(x_1),....,\alpha_{n-1}(x_{n-1}),[y_1,....,y_n]_t\big]_t =$$

$$\sum_{i=1}^{n-1}\big[\alpha_1(y_1),....,\alpha_{i-1}(y_{i-1}),[x_1,....,x_{n-1},y_i]_t,\alpha_i(y_{i+1}),...,\alpha_{n-1}(y_n)\big]_t. \tag{5.1}$$

The deformation is said to be of order k if $[\cdot,...,\cdot]_t = \sum_{i=0}^{k} t^i[\cdot,...,\cdot]_i$ and infinitesimal if $t^2 = 0$.

In terms of elements $x = (x_i)_{1\leq i\leq n-1}$, $y = (y_i)_{1\leq i\leq n-1}\in\mathcal{L}(\mathcal{N})$ and setting $z = y_n$ the above condition reads

$$L_t([x,y]_\alpha)\cdot\alpha_n(z) = L_t(\tilde{\alpha}(x))\cdot\big(L_t(y)\cdot z\big) - L_t(\tilde{\alpha}(y))\cdot\big(L_t(x).z\big) \tag{5.2}$$

where $L_t(x)\cdot z = [x_1,...,x_{n-1},z]_t$ and $\tilde{\alpha}(x) = (\alpha_i(x_i))_{1\leq i\leq n-1}$.
Now let $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ be a multiplicative Hom-Nambu-Lie (i.e. $\alpha_1 = ... = \alpha_n = \alpha$).
Eq. (5.2) implies, keeping only terms linear in t,

$$
\begin{aligned}
&\big[\alpha(x_1),....,\alpha(x_{n-1}),\psi(y_1,....,y_n)\big] + \psi\big(\alpha(x_1),....,\alpha(x_{n-1}),[y_1,....,y_n]\big) \\
= &\sum_{i=1}^{n}\big[\alpha(y_1),....,\alpha(y_{i-1}),\psi(x_1,....,x_{n-1},y_i),\alpha(y_{i+1}),...,\alpha(y_n)\big] \\
+ &\sum_{i=1}^{n}\psi\big(\alpha(y_1),....,\alpha(y_{i-1}),[x_1,....,x_{n-1},y_i],\alpha(y_{i+1}),...,\alpha(y_n)\big).
\end{aligned}
$$

This expression may be read as the 1-cocycle condition $\delta^1\psi = 0$ for the $\mathcal{N}$-valued cochain ψ. In terms of $x,\ y \in \mathcal{L}(\mathcal{N})$ it may be written, (setting again $y_n = z$), as

$$\begin{aligned}\delta^1\psi(x,y,z) &= \psi(\alpha(x), L(y)\cdot z) - \psi(\alpha(y), L(x)\cdot z) - \psi([x_1,x_2]_\alpha, \alpha(z)) \\ &+ L(\alpha(x))\cdot\psi(y,z) - L(\alpha(y))\cdot\psi(x,z) + \big(\psi(x,\quad)\cdot y\big)\bullet_\alpha \alpha(z)\end{aligned} \tag{5.3}$$

where

$$\big(\psi(x,\quad)\cdot y\big)\bullet_\alpha \alpha(z) = \sum_{i=0}^{n-1}[\alpha(y_1),...,\psi(x,y_i),...,\alpha(y_{n-1}),\alpha(z)]. \tag{5.4}$$

Definition 5.2. a p-cochains is an $p+1$-linear map $\varphi : \mathcal{L}(\mathcal{N})\otimes ...\otimes\mathcal{L}(\mathcal{N})\wedge\mathcal{N}\longrightarrow\mathcal{N}$, such that

$$\alpha\circ\varphi(x_1,...,x_p,z) = \varphi(\alpha(x_1),...,\alpha(x_p),\alpha(z)).$$

We denote the set of a p-cochain by $C^p(\mathcal{N},\mathcal{N})$

Definition 5.3. We call, for $p\geq 1$, p-coboundary operator of the multiplicative Hom-Nambu-Lie $(\mathcal{N},[\cdot,...,\cdot],\alpha)$ the linear map $\delta^p : C^p(\mathcal{N},\mathcal{N})\to C^{p+1}(\mathcal{N},\mathcal{N})$ defined by

$$\begin{aligned}\delta^p\psi(x_1,...,x_p,x_{p+1},z) &= \sum_{1\leq i\leq j}^{p+1}(-1)^i\psi\big(\alpha(x_1),...,\widehat{\alpha(x_i)},...,\alpha(x_{j-1}),[x_i,x_j]_\alpha,...,\alpha(x_{p+1}),\alpha(z)\big) \\ &+ \sum_{i=1}^{p+1}(-1)^i\psi\big(\alpha(x_1),...,\widehat{\alpha(x_i)},...,\alpha(x_{p+1}),L(x_i)\cdot z\big) \\ &+ \sum_{i=1}^{p+1}(-1)^{i+1}L(\alpha^p(x_i))\cdot\psi\big(x_1,...,\widehat{x_i},...,x_{p+1},z\big) \\ &+ (-1)^p\big(\psi(x_1,...,x_p,\ )\cdot x_{p+1}\big)\bullet_\alpha\alpha^p(z)\end{aligned} \tag{5.5}$$

where

$$\big(\psi(x_1,...,x_p,\ )\cdot x_{p+1}\big)\bullet_\alpha\alpha^p(z) = \sum_{i=1}^{n-1}[\alpha^p(x_{p+1}^1),...,\psi(x_1,...,x_p,x_{p+1}^i),...,\alpha^p(x_{p+1}^{n-1}),\alpha^p(z)], \tag{5.6}$$

for all $x_i = (x_i^j)_{1\leq j\leq n-1}\in\mathcal{L}(\mathcal{N}),\ 1\leq i\leq p+1,\ z\in\mathcal{N}$ and $\widehat{x}_i$ designed that x_i is omitted.

Proposition 5.4. *Let $\psi\in C^p(\mathcal{N},\mathcal{N})$ be a p-cochain then*

$$\delta^{p+1}\circ\delta^p(\psi) = 0.$$

Proof. Let ψ be a p-cochain, $x_i = (x_i^j)_{1\leq j\leq n-1}\in\mathcal{L}(\mathcal{N}),\ 1\leq i\leq p+2$ and $z\in\mathcal{N}$ we can write δ^p and $\delta^{p+1}\circ\delta^p$ as

$$\delta^p = \delta_1^p + \delta_2^p + \delta_3^p + \delta_4^p,$$

$$\text{and}\qquad \delta^{p+1}\circ\delta^p = \sum_{i,j=1}^{4}\eta_{ij},$$

when $\eta_{ij}=\delta_i^{p+1}\circ\delta_j^p$ and

$$\begin{aligned}
\delta_1^p\psi(x_1,...,x_{p+1},z) &= \sum_{1\le i<j}^{p+1}(-1)^i\psi\big(\alpha(x_1),...,\widehat{x}_i,...,[x_i,x_j]_\alpha,...,\alpha(x_{p+1}),\alpha(z)\big)\\
\delta_2^p\psi(x_1,...,x_{p+1},z) &= \sum_{i=1}^{p+1}(-1)^i\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),L(x_i).z\big)\\
\delta_3^p\psi(x_1,...,x_{p+1},z) &= \sum_{i=1}^{p+1}(-1)^{i+1}L(\alpha^p(x_i))\cdot\psi\big(x_1,...,\widehat{x}_i,...,x_{p+1},z\big)\\
\delta_4^p\psi(x_1,...,x_{p+1},z) &= (-1)^p\big(\psi(x_1,...,x_p,\ )\cdot x_{p+1}\big)\bullet_\alpha\alpha^p(z)
\end{aligned}$$

To simplify the notations we replace $L(x)\cdot z$ by $x\cdot z$.
The proof that $\eta_{11}+\eta_{12}+\eta_{21}+\eta_{22}=0$ is similar to the proof in Proposition 4.4.
On the other hand, we have

$$\begin{aligned}
\star\ \eta_{13}\psi(x_1,...,x_{p+2},z) &= \sum_{1\le i<j<k}^{p+2}\big\{(-1)^{k+i}\alpha^{p+1}(x_k)\cdot\psi(\alpha(x_1),...,\widehat{x}_i,...,[x_i,x_j]_\alpha,...,\widehat{x}_k,...,\alpha(z))\\
&+ (-1)^{j+i}\alpha^{p+1}(x_j)\cdot\psi(\alpha(x_1),...,\widehat{x}_i,...,\widehat{x}_j,...,[x_i,x_k]_\alpha,...,\alpha(z))\\
&+ (-1)^{j+i-1}\alpha^{p+1}(x_i)\cdot\psi(\alpha(x_1),...,\widehat{x}_i,...,\widehat{x}_j,...,[x_j,x_k]_\alpha,...,\alpha(z))\big\}
\end{aligned}$$

$$\begin{aligned}
\star\ \eta_{31}\psi(x_1,...,x_{p+2},z) &= -\eta_{13}\psi(x_1,...,x_{p+2},z)\\
&+ \sum_{1\le i<j}^{p+2}(-1)^{i+j}\alpha^p([x_i,x_j]_\alpha)\cdot\alpha\big(\psi(x_1,...,\widehat{x}_i,...,\widehat{x}_j,...,z)\big)
\end{aligned}$$

$$\begin{aligned}
\star\ \eta_{33}\psi(x_1,...,x_{p+2},z) &= \sum_{1\le i<j}^{p+2}\Big\{(-1)^{i+j}\alpha^{p+1}(x_i)\cdot\big(\alpha^p(x_j).\big(\psi(x_1,...,\widehat{x}_i,...,\widehat{x}_j,...,z)\big)\big)\\
&+ (-1)^{i+j-1}\alpha^{p+1}(x_j)\cdot\big(\alpha^p(x_i)\cdot\big(\psi(x_1,...,\widehat{x}_i,...,\widehat{x}_j,...,z)\big)\big)\Big\}
\end{aligned}$$

Then, applying Lemma 3.1 to $\alpha^p(x_i)\in\mathcal{L}(\mathcal{N})$, $\alpha^p(x_j)\in\mathcal{L}(\mathcal{N})$ et $\psi(x_1,...,\widehat{x}_i,...,\widehat{x}_j,...,z)\in\mathcal{N}$, we have

$$\eta_{13}+\eta_{33}+\eta_{31}=0.$$

by the same calculation, we can prove that

$$\eta_{23}+\eta_{32}=0.$$

$$\begin{aligned}
&\star\ \eta_{14}\psi(x_1,...,x_{p+2},z)\\
= &\ (-1)^p\sum_{1\le i<j}^{p+1}(-1)^i\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x_{p+2}^1),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,[x_i,x_j]_\alpha,...,\alpha(x_p),\alpha(x_{p+1}^k)\big),...,\alpha^{p+1}(z)\big]\\
+ &\ (-1)^p\sum_{i=1}^{p+1}(-1)^i\sum_{k,l=1;k\ne l}^{n-1}\big[\alpha^{p+1}(x_{p+2}^1),...,\alpha^p(x_i\cdot x_{p+2}^l),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x_{p+2}^k)\big),...,\alpha^{p+1}(z)\big]\\
+ &\ (-1)^p\sum_{i=1}^{p+1}(-1)^i\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x_{p+2}^1),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),x_i\cdot x_{p+2}^k\big),...,\alpha^{p+1}(z)\big].
\end{aligned}$$

The first term in η_{14} is equal to $-\eta_{41}$, hence

$$\begin{aligned}
&\star\ (\eta_{14}+\eta_{41})\psi(x_1,...,x_{p+2},z)\\
=\ &\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k,l=1;k\neq l}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,\alpha^p(x_i\cdot x^l_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^{p+1}(z)\big]\\
+\ &\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),x_i\cdot x^k_{p+2}\big),...,\alpha^{p+1}(z)\big]
\end{aligned}$$

and

$$\begin{aligned}
&\star\ \eta_{24}\psi(x_1,...,x_{p+2},z)\\
=\ &\sum_{i=1}^{p+1}\sum_{k=1}^{n-1}(-1)^{p+i}\big[\alpha^{p+1}(x^1_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^p(x_i\cdot z)\big]\\
+\ &\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+1}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_p),\alpha(x^k_{p+1})\big),...,\alpha^p(x_{p+2}\cdot z)\big]
\end{aligned}$$

$$\begin{aligned}
&\star\ \eta_{42}\psi(x_1,...,x_{p+2},z)\\
=\ &(-1)^{p+1}\sum_{i=1}^{p+1}(-1)^{i}\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),x_i\cdot x^k_{p+2}\big),...,\alpha^{p+1}(z)\big].
\end{aligned}$$

Hence, $-\eta_{42}$ and the second term of $(\eta_{14}+\eta_{41})$ are equal.

Using the Hom-Nambu identity for any integers $1\leq i\leq p+1$ et $1\leq k\leq n-1$

$$\begin{aligned}
&\alpha^{p+1}(x_i)\cdot\big[\alpha^p(x^1_{p+2}),...,\psi\big(x_1,...,\widehat{x}_i,...,x_{p+1},x^k_{p+2}\big),...,\alpha^p(z)\big]\\
=\ &\sum_{l=1;l\neq k}^{n-1}\Big\{\big[\alpha^{p+1}(x^1_{p+2}),...,\alpha^p(x_i\cdot x^l_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^{p+1}(z)\big]\Big\}\\
&+\big[\alpha^{p+1}(x^1_{p+2}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^p(x_i\cdot z)\big]\\
&+\big[\alpha^{p+1}(x^1_{p+1}),...,\alpha^p(x_i)\cdot\psi\big(x_1,...,\widehat{x}_i,...,x_{p+1},x^k_{p+2}\big),...,\alpha^{p+1}(z)\big]
\end{aligned}$$

when we add the four terms η_{14}, η_{41}, η_{24} and η_{42}, we have the following expression

$$\begin{aligned}
&(\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42})\psi(x_1,...,x_{p+2},z)\\
=\ &\sum_{i=1}^{p+1}(-1)^{i+p}\sum_{l=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,\alpha^p(x_i)\cdot\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^{p+1}(z)\big]\\
+\ &(-1)^{p-1}\sum_{i=1}^{p+1}(-1)^{i}\sum_{k=1}^{n-1}\alpha^{p+1}(x_i)\cdot\big[\alpha^p(x^1_{p+2}),...,\psi\big(x_1,...,\widehat{x}_i,...,x_{p+1},x^k_{p+2}\big),...,\alpha^p(z)\big]\\
+\ &\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+1}),...,\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_p),\alpha(x^k_{p+1})\big),...,\alpha^p(x_{p+2}\cdot z)\big]
\end{aligned}$$

and

$$\begin{aligned}
&\star\ \eta_{43}\psi(x_1,...,x_{p+2},z)\\
=\ &\sum_{i=1}^{p+1}(-1)^{p+i}\sum_{k=1}^{n-1}\alpha^{p+1}(x_i)\cdot\big[\alpha^p(x^1_{p+2}),...,\psi\big(x_1,...,\widehat{x}_i,...,x_{p+1},x^k_{p+2}\big),...,\alpha^p(z)\big]\\
-\ &\sum_{k=1}^{n-1}\alpha^{p+1}(x_{p+2})\cdot\big[\alpha^p(x^1_{p+1}),...,\psi\big(x_1,...,x_p,x^k_{p+1}\big),...,\alpha^p(z)\big],
\end{aligned}$$

$$\begin{aligned}&\eta_{34}\psi(x_1,...,x_{p+2},z)\\ =&\sum_{i=1}^{p+1}(-1)^{i+p+1}\sum_{l=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,\alpha^p(x_i)\cdot\psi\big(\alpha(x_1),...,\widehat{x}_i,...,\alpha(x_{p+1}),\alpha(x^k_{p+2})\big),...,\alpha^{p+1}(z)\big].\end{aligned}$$

Hence

$$\begin{aligned}&(\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42}+\eta_{34}+\eta_{43})\psi(x_1,...,x_{p+2},z)=-\eta_{44}\psi(x_1,...,x_{p+2},z)\\ =&-\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\big[\alpha^{p+1}(x^1_{p+2}),...,[\alpha^p(x^1_{p+1}),...,\psi\big(x_1,...,x_p,x^k_{p+1}\big),...,\alpha^p(x^i_{p+2})],...,\alpha^{p+1}(x^{n-1}_{p+2}),\alpha^{p+1}(z)\big]\\ =&-\sum_{k=1}^{n-1}\alpha^{p+1}(x_{p+2})\cdot[\alpha^p(x^1_{p+1}),...,\psi\big(x_1,...,x_p,x^k_{p+1}\big),...,\alpha^p(z)]\\ +&\sum_{k=1}^{n-1}[\alpha^{p+1}(x^1_{p+1}),...,\psi\big(\alpha(x_1),...,\alpha(x_p),\alpha(x^k_{p+1})\big),...,\alpha^p(x_{p+2}\cdot z)].\end{aligned}$$

Then, we have

$$\eta_{14}+\eta_{41}+\eta_{24}+\eta_{42}+\eta_{34}+\eta_{43}+\eta_{44}=0,$$

which ends the proof. □

Definition 5.5. The space of p-cocycles is defined by

$$Z^p(\mathcal{N},\mathcal{N})=\{\varphi\in C^p(\mathcal{N},\mathcal{N}):\delta^p\varphi=0\},$$

and the space of p-coboundaries is defined by

$$B^p(\mathcal{N},\mathcal{N})=\{\psi=\delta^{p-1}\varphi:\varphi\in C^{p-1}(\mathcal{N},\mathcal{N})\}.$$

Lemma 5.6. $B^p(\mathcal{N},\mathcal{N})\subset Z^p(\mathcal{N},\mathcal{N})$

Definition 5.7. We call the p^{th}-cohomology group the quotient

$$H^p(\mathcal{N},\mathcal{N})=\frac{Z^p(\mathcal{N},\mathcal{N})}{B^p(\mathcal{N},\mathcal{N})}.$$

6. Cohomology of n-ary Hom-algebras induced by cohomology of Hom-Leibniz algebras

6.1. **Cohomology of ternary Hom-Nambu algebras induced by cohomology of Hom-Leibniz algebras.** In this section we extend to ternary multiplicative Hom-Nambu-Lie algebras the Takhtajan's construction of a cohomology of ternary Nambu-Lie algebras starting from Chevalley-Eilenberg cohomology of binary Lie algebras, (see [12, 31, 32]). The cohomology of multiplicative Hom-Lie algebras was introduced in [2] and independently in [30].
The cohomology complex for Leibniz algebras was defined by Loday-Pirashvili in [18]. We extend it to Hom-Leibniz algebras as follows.
Let $(A,[\cdot,\cdot],\alpha)$ be a Hom-Leibniz algebras and $\mathcal{C}_{\mathcal{L}}(A,A)$ be the set of cochains $\mathcal{C}^p_{\mathcal{L}}(A,A)=Hom(\otimes^pA,A)$ for $n\geq 1$. We set $\mathcal{C}^0_{\mathcal{L}}(A,A)=A$. We define a coboundary operator d by $d\varphi(a)=-[\varphi,a]$ when $\varphi\in\mathcal{C}^0_{\mathcal{L}}(A,A)$ and for $p\geq 1$, $\varphi\in\mathcal{C}^p_{\mathcal{L}}(A,A)$, $a_1,\cdots,a_{p+1}\in A$

(6.1)

$$\begin{aligned}d^p\varphi(a_1,\cdots,a_{p+1})=&\sum_{k=1}^{p}(-1)^{k-1}\big[\alpha^{p-1}(a_k),\varphi(a_1,\cdots,\widehat{a_k},\cdots,a_{p+1})\big]\\ &+(-1)^{p+1}\big[\varphi(a_1\otimes\cdots\otimes a_p),\alpha^{p-1}(a_{p+1})\big]\\ &+\sum_{1\leq k<j}^{p+1}(-1)^k\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k,a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1}))\end{aligned}$$

Notice that we recover the classical case when $\alpha = id$.

We aim now to derive the cohomology of a ternary Hom-Nambu algebra from the cohomology of Hom-leibniz algebra following the procedure described for ternary Nambu algebra in [12].

Let $(\mathcal{N}, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative ternary Hom-Nambu-Lie algebra. Using Proposition 3.2 the triple $(\mathcal{L}(\mathcal{N}) = \mathcal{N} \otimes \mathcal{N}, [\cdot,\cdot]_\alpha, \alpha)$ where the bracket is defined for $x = x_1 \otimes x_2$ and $y = y_1 \otimes y_2$ by

$$[x, y] = [x_1, x_2, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x_1, x_2, y_2], \tag{6.2}$$

is a Hom-Leibniz algebra.

Theorem 6.1. *Let $(\mathcal{N}, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative ternary Hom-Nambu-Lie algebra and $\mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) = Hom(\otimes^{2p+1}\mathcal{N},\mathcal{N})$ for $n \geq 1$ be the cochains. Let $\Delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L},\mathcal{L})$ be the linear map defined for $p = 0$ by*

$$\Delta\varphi(x_1 \otimes x_2) = x_1 \otimes \varphi(x_2) + \varphi(x_1) \otimes x_2$$

and for $p > 0$

$$(\Delta\varphi)(a_1, \cdots, a_{p+1}) = \alpha^{p-1}(x_{2p+1}) \otimes \varphi(a_1, \cdots, a_p \otimes x_{2p+2}) + \varphi(a_1, \cdots, a_p \otimes x_{2p+1}) \otimes \alpha^{p-1}(x_{2p+2}),$$

where we set $a_j = x_{2j-1} \otimes x_{2j}$.
Then there exists a cohomology complex $(\mathcal{C}^\bullet_{\mathcal{N}}(\mathcal{N},\mathcal{N}), \delta)$ for ternary Hom-Nambu-Lie algebras such that

$$d \circ \Delta = \Delta \circ \delta.$$

The coboundary map $\delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ is defined for $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ by

(6.3)
$$\begin{aligned}\delta\varphi(x_1 \otimes \cdots \otimes x_{2p+1}) = &\sum_{j=1}^{p}\sum_{k=2j+1}^{2p+1} (-1)^j \varphi(\alpha(x_1) \otimes \cdots \otimes [x_{2j-1}, x_{2j}, x_k] \otimes \cdots \otimes \alpha(x_{2p+1}))+\\ &\sum_{k=1}^{p} (-1)^{k-1}[\alpha^{p-1}(x_{2k-1}), \alpha^{p-1}(x_{2k}), \varphi(x_1 \otimes \cdots \otimes \widehat{x_{2k-1}} \otimes \widehat{x_{2k}} \otimes \cdots \otimes x_{2p+1})]+\\ &(-1)^{n+1}[\alpha^{p-1}(x_{2p-1}), \varphi(x_1 \otimes \cdots \otimes x_{2p-2} \otimes x_{2p}), \alpha^{p-1}(x_{2p+1})]+\\ &(-1)^{p+1}[\varphi(x_1 \otimes \cdots \otimes x_{2p-1}), \alpha^{p-1}(x_{2p}), \alpha^{p-1}(x_{2p+1})]\end{aligned}$$

Proof. The proof is a particular case of Theorem 6.3 proof. □

Remark 6.2. The theorem shows that one may derive the cohomology complex of ternary Hom-Nambu-Lie algebras from the cohomology complex of Hom-Leibniz algebras.

6.2. **Cohomology of n-ary Hom-Nambu-Lie algebras induced by cohomology of Hom-Leibniz algebras.** We generalize in this section the result of the previous section to n-ary Hom-Nambu-Lie algebras.

Let $(\mathcal{N}, [\cdot, ..., \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra and the triple $(\mathcal{L}(\mathcal{N}) = \mathcal{N}^{\otimes n-1}, [\cdot,\cdot]_\alpha, \alpha)$ be the Hom-Leibniz algebra associates to $\mathcal{N}$ where the bracket is defined in (3.4).

Theorem 6.3. *Let $(\mathcal{N}, [\cdot, ..., \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra and $\mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) = Hom(\otimes^p\mathcal{L}(\mathcal{N}) \otimes \mathcal{N},\mathcal{N})$ for $p \geq 1$ be the cochains. Let $\Delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L},\mathcal{L})$ be the linear map defined for $p = 0$ by*

$$\Delta\varphi(x_1 \otimes \cdots \otimes x_{n-1}) = \sum_{i=0}^{n-1} x_1 \otimes \cdots \otimes \varphi(x_i) \otimes \cdots \otimes x_{n-1} \tag{6.4}$$

and for $p > 0$ by

$$(\Delta\varphi)(a_1, \cdots, a_{p+1}) = \sum_{i=1}^{n-1} \alpha^{p-1}(x^1_{p+1}) \otimes \cdots \otimes \varphi(a_1, \cdots, a_{n-1} \otimes x^i_{p+1}) \otimes \cdots \otimes \alpha^{n-1}(x^{n-1}_{p+1}), \tag{6.5}$$

where we set $a_j = x_j^1 \otimes \cdots \otimes x_j^{n-1}$.
Then there exists a cohomology complex $(\mathcal{C}^{\bullet}_{\mathcal{N}}(\mathcal{N},\mathcal{N}),\delta)$ *for n-ary Hom-Nambu-Lie algebras such that*

$$d \circ \Delta = \Delta \circ \delta.$$

The coboundary map $\delta : \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N}) \to \mathcal{C}^{p+1}_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ *is defined for* $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ *by*

$$\begin{aligned}
\delta\varphi(a_1,...,a_p,a_{p+1},x) &= \sum_{1\le i\le j}^{p+1} (-1)^i \varphi\big(\alpha(a_1),...,\widehat{\alpha(a_i)},...,\alpha(a_{j-1}),[a_i,a_j]_\alpha,...,\alpha(a_{p+1}),\alpha(x)\big)\\
&+ \sum_{i=1}^{p+1}(-1)^i\varphi\big(\alpha(a_1),...,\widehat{\alpha(a_i)},...,\alpha(a_{p+1}),L(a_i).x\big)\\
&+ \sum_{i=1}^{p+1}(-1)^{i+1}L(\alpha^p(a_i))\cdot\varphi\big(a_1,...,\widehat{a_i},...,a_{p+1},x\big)\\
&+ (-1)^p\big(\varphi(a_1,...,a_p,\ )\cdot a_{p+1}\big)\bullet_\alpha \alpha^p(x),
\end{aligned}$$

where

$$\big(\varphi(a_1,...,a_p,\ )\cdot a_{p+1}\big)\bullet_\alpha \alpha^p(x) = \sum_{i=1}^{n-1}[\alpha^p(x^1_{p+1}),...,\varphi(a_1,...,a_p,x^i_{p+1}),...,\alpha^p(x^{n-1}_{p+1}),\alpha^p(x)].$$

for all $a_i \in \mathcal{L}(\mathcal{N})$, $x \in \mathcal{N}$.

Proof. Let $\varphi \in \mathcal{C}^p_{\mathcal{N}}(\mathcal{N},\mathcal{N})$ and $(a_1\cdots a_{p+1}) \in \mathcal{L}$ where $a_j = x_1^j \otimes \cdots \otimes x^j_{n-1}$.
Then $\Delta\varphi \in \mathcal{C}^{p+1}_{\mathcal{L}}(\mathcal{L},\mathcal{L})$ and using (6.1) we can to write $d = d_1 + d_2 + d_3$, where

$$\begin{aligned}
d_1\varphi(a_1,\cdots,a_{p+1}) &= \sum_{k=1}^{p}(-1)^{k-1}\big[\alpha^{p-1}(a_k),\varphi(a_1,\cdots,\widehat{a_k},\cdots,a_{p+1})\big]\\
d_2\varphi(a_1,\cdots,a_{p+1}) &= (-1)^{p+1}\big[\varphi(a_1\otimes\cdots\otimes a_p),\alpha^{p-1}(a_{p+1})\big]\\
d_3\varphi(a_1,\cdots,a_{p+1}) &= \sum_{1\le k<j}^{p+1}(-1)^k\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k,a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1}))
\end{aligned}$$

By (6.5) we have

$$
\begin{aligned}
&d_1 \circ \Delta\varphi(a_1,\cdots,a_{p+1})\\
&=\sum_{k=1}^{p}(-1)^{k-1}\big[\alpha^{p-1}(a_k),\Delta\varphi(a_1,\cdots,\widehat{a_k},\cdots,a_{p+1})\big]\\
&=\sum_{k=1}^{p}(-1)^{k-1}\sum_{i=1}^{n-1}\big[\alpha^{p-1}(a_k),\alpha^{p-1}(x^1_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes\alpha^{p-1}(x^{n-1}_{p+1})\big]\\
&=\sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})\\
&+\sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})\\
&+\sum_{k=1}^{p}(-1)^{k-1}\sum_{i=1}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})\\
&=\sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})\\
&+\sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})\\
&+\Delta\circ\delta_3\circ\varphi(a_1,\cdots,a_{p+1})\\
&=\Lambda_1+\Lambda_2+\Delta\circ\delta_3\circ\varphi(a_1,\cdots,a_{p+1})
\end{aligned}
$$

where

$$
\Lambda_1=\sum_{k=1}^{p}(-1)^{k-1}\sum_{i>j}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})
$$

$$
\Lambda_2=\sum_{k=1}^{p}(-1)^{k-1}\sum_{j>i}^{n-1}\alpha^p(x^1_{p+1})\otimes\cdots\otimes\varphi(a_1,\cdots,\widehat{a_k},\cdots,x^i_{p+1})\otimes\cdots\otimes L(\alpha^{p-1}(x_k)).\alpha^{p-1}(x^j_{p+1})\otimes\cdots\otimes\alpha^p(x^{n-1}_{p+1})
$$

Similarly we can prove that

$$
d_2\circ\Delta\varphi(a_1,\cdots,a_{p+1}) \ =\Delta\circ\delta_4\varphi(a_1,\cdots,a_{p+1})
$$

and

$$
\begin{aligned}
&d_3\Delta\circ\varphi(a_1,\cdots,a_{p+1})\\
&=\sum_{1\leq k<j}^{p}(-1)^k\Delta\circ\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_{j-1})\otimes[a_k,a_j]\otimes\alpha(a_{j+1})\otimes\cdots\otimes\alpha(a_{p+1}))\\
&+\sum_{k=1}^{p+1}(-1)^k\varphi(\alpha(a_1)\otimes\cdots\otimes\widehat{a_k}\otimes\cdots\otimes\alpha(a_p)\otimes[a_k,a_{p+1}])\\
&=\Delta\circ\delta_1\varphi(a_1,\cdots,a_{p+1})+\Delta\circ\delta_2\varphi(a_1,\cdots,a_{p+1})\\
&+\Lambda'_1+\Lambda'_2
\end{aligned}
$$

where $\Lambda'_1=-\Lambda_1$ and $\Lambda'_2=-\Lambda_2$.

Finally we have

$$
d\circ\Delta=d_1\circ\Delta+d_2\circ\Delta+d_3\circ\Delta=\Delta\circ\delta_3+\Delta\circ\delta_4+\Delta\circ\delta_1+\Delta\circ\delta_2=\Delta\circ\delta
$$

where $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$ as defined in Proof 5.4. □

Remark 6.4. If $d^2 = 0$, then $\delta^2 = 0$.
In fact,we have $d \circ \Delta = \Delta \circ \delta$, then

$$\Delta \circ \delta^2 = \Delta \circ \delta \circ \delta = d \circ \Delta \circ \delta = d \circ .d \circ \Delta = d^2 \circ \Delta = 0.$$

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